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CONSTRUCTION OF THE VALUE FUNCTION IN A GAME OF APPROACH WITH SEVERAL PURSUERS†

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A differential game of approach of one evader with n dynamic pursuers is investigated. All the players have simple motions. The velocities of the players and the time of the game are limited. The case of a game with both similar arbitrary pursuers and pursuers of different types whose velocities exceed that of the evader are considered. The payoff is taken to be the distance between the evader and the closest pursuer at the instant when the game terminates. The formalization used is the same as that employed in [1, 2]. The method used in [3, 4] of solving problems of the approach of two pursuers with a single evader in a plane is extended to the solution of the problem of the approach of n pursuers with a single evader in space R^n. The value function of the game is constructed not only in a regular region [5], but also in a separate singular manifold. A programmed maximin function is introduced, the space of the initial positions is decomposed into regions, and the u-stability of the function over the whole of the space is proved. An example of a game of pursuit of "three after one" in a three-dimensional Euclidean space is given, and the optimal trajectories and the level surfaces of the value of the game are constructed.

1. FORMULATION OF THE PROBLEM

THE DYNAMICS of pursuers, combined in a coalition, and the dynamics of an evader are described by the following equations

$$x_i^{\bullet}(t)=u_i(t),\;\; x_i(0)=x_i^0,\;\; \|u_i\|\leqslant \rho_i,\rho_i\geqslant 0,\;\; i=1,n \tag{1.1}$$

$$y^{\bullet}(t)=v(t),\;\; y(0)=y^0,\;\; \|v\|\leqslant\sigma,\;\; \sigma\geqslant 0,\;\; t\in[0,T],\;\; T<\infty \tag{1.2}$$

Here x_i and y are the positions of the pursuers and the evader, respectively, in an n-dimensional space, the control $u_i(\cdot)$ is formed during the motion of the position strategy U_i: $T\times R^n \Rightarrow B_{\rho_i}(0)$, the control of the evader is chosen as an arbitrary function of time from the class V_t of measured functions $v(\cdot)$: $T\Rightarrow B_\sigma(0)$, where $B_\rho(0)$ and $B_\sigma(0)$ are n-dimensional closed spheres of radius ρ and σ with centre at the origin of coordinates, and $\|(\cdot)\|$ is the Euclidean norm. The payoff function is the minimum distance between the evader and the pursuers at the instant when the game terminates

$$\Phi(x_i,y)=\min_i d(x_i(T),y(T)) \tag{1.3}$$

where $d(x_i(T), y(T))$ is the Euclidean distance between $x_i(T)$ and $y(T)$.

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The aim of the pursuers is to minimize the payoff function while the aim of the evader is to maximize it.

It is required to construct the function of the guaranteed result.

2. THE PROGRAMMED MAXIMIN FUNCTION

The domain of attainability of the players $x_i(t)$ and $y(t)$ up to the instant T are n-dimensional spheres $G_t^i = B_{\rho_i(T-t)}(x_i(t))$ and $G_t = B_{\sigma(T-t)}(y(t))$ of radii $\rho_i(T-t)$ and $\sigma(T-t)$ with centres at $x_i(t)$ and $y(t)$, respectively.

We define the programmed maximin function as follows:

$$\gamma(t, x_i(t), y(t)) = \max_{y \in G_t} \min_i \min_{x_i \in G_t^i} d(x_i, y) \tag{2.1}$$

When $t = T$ the programmed maximin function is identical with the payoff function (1.3).

Suppose W_k is a convex linear hull of the positions k of the pursuers at an arbitrary instant of time, and $W_k{}^*$ is that of the k pursuers and the evader at the same instant

$$\begin{aligned} &W_k = \{w\colon w = \sum_{i=1}^{k} \alpha_i x_i,\ \sum_{i=1}^{k} \alpha_i = 1,\ i \in J_k\} \\ &W_k^* = \{w^*: w^* = \alpha_1 w + \alpha_2 y,\ \alpha_1 + \alpha_2 = 1,\ w \in W_k\} \\ &J_k = \{j_1, j_2, \ldots, j_k\},\ J_k \in I,\ J_k \neq \Phi \end{aligned} \tag{2.2}$$

where J_k is a non-empty manifold J_k of the set of indices $I = \{1, 2, \ldots, n\}$.

It is convenient to consider only those J_k which form W_k of dimensions $k-1$; then, L_k is a subspace equidistant (when $\rho_1 = \rho_2 = \ldots = \rho_n$) from the k pursuers and has dimensions $n-k+1$

$$\begin{aligned} &L_k = \{l\colon d(l, x_p) = d(l, x_q),\ p \neq q, p, q \in J_k\} \\ &\dim L_k = n - \dim W_k \end{aligned} \tag{2.3}$$

When there are different limitations on the velocities of the pursuers, L_k is a subspace orthogonal to W_k and containing all extremal aiming points.

In order for the problem of the pursuit of n pursuers after a single evader in the problem of the pursuit of $k < n$ after one to be non-degenerate, it is necessary that $\dim W_n = n-1$, $\dim L_k = 1$.

A minimum and a maximum of the programmed maximin function is reached at the boundaries of the regions G_t^i and G_t.

We will call the point $y^*(t)$ the point of extremal aiming if the programmed maximin function reaches a maximum at this point. The distance from the extremal aiming point to an arbitrary pursuer is made up of the length of the radius of its domain of attainability and a certain constant which is common for all the pursuers. By choosing the subspace J_k from I such that a minimum is reached at $x_i(t)$, $i \in J_k$. we can write (2.1) in the following form

$$\begin{aligned} &\gamma(t, x_i(t), y(t)) = \max_{y \in G_t} \min_i d(x_i(t), y) - \rho_i(T-t) = d(x_i(t), y^*(t)) - \rho_i(T-t) \\ &i \in J_k,\ x_i(t) \in W_k,\ k \leqslant n \end{aligned} \tag{2.4}$$

3. POSSIBLE CASES OF THE MUTUAL POSITION OF THE PLAYERS

At each instant of time t the evader $y(t)$ may be in one of several regions D_k $(k=1, 2, \ldots, n)$ relative to the pursuers $x_i(t)$, where the problem degenerates into a game of pursuit by only k players.

We will define the region to which $y(t)$ belongs of the region D_k by the index 1, and the singular manifold in the region D_k by the index 2.

Index 1. In a game of pursuit by n pursuers at the instant of time t we have a game of pursuit by k pursuers, where $k \leqslant n$, if J_k and J_{n-k} exist such that

$$\gamma(t, x_i(t), \qquad y(t)) = \min_{i \in J_k} d(x_i, y^*(t)) < \min_{i \in J_{n-k}} d(x_i, y^*(t))$$

where $J_{n-k} = I \setminus J_k$, and we will write $y(t) \in D_k$.

We note one property of the regions $D_k : D_l \cap D_m \neq \phi,\ k,\ l < n$.

Index 2. In a game of pursuit by n pursuers at the instant of time t, $y(t)$ belongs to the singular manifold S_k if

1. $y(t) \in D_k$,

2. for a given J_k the subspaces W_k and $W_k{}^*$ (2.2) are defined and $\dim W_k = \dim W_k{}^*$.

For each region D_k a singular manifold exists in which the piecewise-smooth programmed maximin function can have a discontinuity.

When $k=n$ and $y(t) \in S_n$ there are two extremal aiming points

$$d(y_1^*(t), W_n) = d(y_2^*(t), W_n), \quad \{y_1^*(t), y_2^*(t)\} = L_n \cap D_t$$

unlike the regular case where the extremal aiming point is unique.

When $k \leqslant n$ and $y(t) \in S_k$ the intersection of L_k and D_t is the section of an n-dimensional sphere with a subspace of dimensions $\dim L_k$ and the power of the set of extremal aiming points is a continuum.

4. AUXILIARY ASSERTIONS

We will divide the interval $[0, T]$ into N parts. We have $y(t_i) = y(t_{i-1}) + v_i \Delta t$, where $[0, T] = {}_i U[t_i t_{i+1}]$, $v_i \in [t_i t_{i+1}]$, $v_i = \text{const}$ and Δt is the diameter of the division.

Consider the ith interval; we can put $t_i = 0$, $t_{i+1} = t$.

We will prove the following Lemmas 1–3 for the case when $k \geqslant 2$, $y \in S_k$, by considering the evader y in each subspace, defined by L_n and x_i for the time interval $[0, t]$.

Suppose $r = d(x(0), y^*(0))$, b and c are the lengths of the projections $[x_i(0), y^*(0)]$ and $[y(0), y^*(0)]$, respectively, onto $(x_i(0), y(0))$.

It can be shown that Lemmas 1 and 2, which apply to the proof of the u-stability of the programmed maximin function (2.4), hold.

Lemma 1. When $v \neq 0$, $v \in S_k$ for any ρ, σ if $r > \sigma T$, the following inequality is satisfied

$$(r - \rho t)^2 - (b - \rho t)^2 \geqslant (\sigma T - \sigma t)^2 - (c - \sigma t)^2 \quad (t < \min(b/\rho, c/\rho))$$

Here, and also in Lemmas 2, 3, and 4, the subscript i on p is omitted.

When $t \geqslant \min(b/\rho, d\rho)$ an instant of time exists when the trajectory of motion of the evader intersects the subspace L_k (2.3) and the position lies in a singular manifold. Taking this instant of time as the initial instant we will use the following lemma.

Lemma 2. When $v = 0$, $v \in S_k$ for any ρ, σ if $r > \sigma T$, the following inequality is satisfied

$$(r-\rho t)^2-(b-\rho t)^2 \geqslant (\sigma T-\sigma t)^2 (t<b/\rho)$$

When $t \geqslant b/\rho$, the instant when the trajectory of the pursuer intersects the subspace L_k will be taken as the initial instant and we will use Lemma 2.

The geometrical meaning of the above lemmas is as follows. If the evader chooses non-zero and zero controls, which do not reduce to a position with a singular manifold, we obtain non-zero controls of the pursuers which retain a position on the singular manifold such that the programmed maximin function does not increase.

We will assume that the OX axis passing through $x_i(0)$ is perpendicular to L_k.

We will denote by α and β the angles between the controls u and v and OX. We will prove the following lemma: for a position lying on the singular manifold and any controls of the evader, the controls of the pursuers are obtained such that the programmed maximin function does not increase. The position can then both converge with the singular manifold and remain on it.

Lemma 3. For any ρ, σ, β we obtain a_i such that $d(x_i(t), y(t)) \leqslant d(x_i(0), y(0)) - \rho t$ when $y(0) \in S_k$.

The choice of α_i such that

$$\alpha_i = \begin{cases} \arcsin(\sigma\rho^{-1}\sin\beta), & \text{if } \sigma\rho^{-1}\sin\beta \leqslant 1 \\ \alpha_i^*, & \text{if } \sigma\rho^{-1}\sin\beta > 1 \end{cases} \tag{4.1}$$

proves the lemma.

Lemma 4. Suppose $y(t) \in D_k$, $B_{\sigma(T-t)}(y(t)) \cap L_k = \{y^*(t)\} \neq \phi$ and $a \geqslant 0$ is a number such that $\{y^*(t)\} \subset \cup_i B_{a+\rho(T-t)}(x_i(t))$. Then $B_{\sigma(T-t)}(y(t)) \subset \cup_i B_{a+\rho(T-t)}(x_i(t))$.

Lemma 4 reflects the fact that if the region reached by the evader, which lies in the region D_k, intersects the subspace L_k in a certain set of points, it is sufficient to cover only this set with spheres with centres coinciding with the positions of the pursuers, in order to cover the whole region that the evader reaches.

5. THE PROPERTY OF u-STABILITY OF THE PROGRAMMED MAXIMIN FUNCTION

For $i=2$ the programmed maximin function is identical with the value function of the game [3–5]. Suppose that for a game of $n-1$ pursuit ($n \geqslant 3$) the programmed maximin function (2.1) is identical with the value function of the game. Then in Assertions 1 and 2 we will prove that the programmed maximin function is the value of the game of pursuit of "n after one".

Assertion 1. The programmed maximin function is u-stable in the singular manifold.

Proof. Consider the time interval $[0, t]$. Suppose $y(0) \in S_n$, $v = \text{const}$. By our assumption, for all the regions D_k $(k \leqslant n-1)$ we obtain the value function of the game identical with (2.1). We will introduce a fictitious evader, obtained from the actual evader by rotating about L_n by the angle between y and x_i in $W_n{}^*$ and lying in a plane defined by x_i and L_n. The extremal aiming point of the actual evader is identical with the extremal aiming point of the fictitious evader.

The equation of the ith pursuer is constructed from the fictitious evader using Lemmas 1–3, and it is also suitable for the actual evader since information only on the extremal aiming point is used.

If $y(T) \in D_n$, then by Lemma 4 at the instant t the following covering is carried out

$$B_{\sigma(T-t)}(y(t)) \in \bigcup_{i=1}^{n} B_{a+\rho_i(T-t)}(x_i(t)) \tag{5.1}$$

under the condition that, at the initial instant of time, the following condition has been satisfied

$$B_{\sigma T}(y(0)) \in \bigcup_{i=1}^{n} B_{a+\rho_i T}(x_i(0))$$

If in the interval $[0, t]$ there is an instant t_1 such that $y(T) \in D_{n-1}$, then by splitting the interval into subintervals $[0, t]=[0, t_1]\cup[t_1, t]$ we can choose in the first subinterval the controls of the pursuers corresponding to the game "n after one", and in the second game "$n-1$ after one". The covering (5.1), which is carried out at the instant t_1, also occurs at the instant t for $y(t) \in D_{n-1}$ (similarly for $y(t) \in D_k$, $k < n-1$), which also denotes the u-stability of the programmed maximin function. Putting $t=T$ in (5.1) we note that the value function of the game is identical with the constant a.

Assertion 2. The programmed maximin function is u-stable over all the space of the game.

In order to prove this, one need only check the u-stability of the programmed maximin function in the regular region. However, this case can easily be reduced to Assertion 1 by introducing a fictitious evader. For any actual evader one can obtain a fictitious evader which satisfies the following requirements:

1. the fictitious evader is in the singular manifold at the initial instant of time;

2. the extremal aiming point of the fictitious and actual evaders are identical at the initial and final instants of time in the interval $[0, t]$;

3. the fictitious evader is displaced by not more than ρt after a time t.

Using Assertion 1, we obtain the u-stability of the programmed maximin function in the regular region and in the whole of space.

Thus, the programmed maximin function is identical with the value function of the game over the whole space, since the programmed maximin function possesses the property of v-stability by virtue of the linearity of system (1.1) and (1.2).

Notes. 1. The condition $r > \sigma T$ of Lemmas 1–3 must be satisfied for all x_i, which only occurs for similar arbitrary pursuers or different types of pursuers which exceed the velocity of the evader.

2. When there are different constraints on the velocity of the pursuers it is necessary to add the following: if at the initial instant of time the exact covering (5.1) is satisfied, then at the following instant of time, for non-optimal motion of the evader (i.e. not at the extremal aiming point), for example, along the singular manifold, the combination of spheres with centres at x_i covers a sphere that is strictly greater than the region which the evader reaches. In this case, the interval $[0, t]$ can be divided into smaller intervals in each of which we assume the extremal aiming point to be fixed, while the covering (5.1) is exact and Lemmas 1–4 are used.

3. The value function of the game is positive, but if at the initial instant of time $r = \rho T$ for pursuers with an excessive speed, the value function of the game is zero, i.e. point capture occurs.

Example. Consider a differential game of group pursuit of "three after one" in a three-dimensional Euclidean space.

Functional (1.3) takes the form

$$\Phi(x_i) = \min_i \Phi_i(x_i), \quad i = 1, 2, 3$$

$$\Phi_i(x_i) = ((x_i^x - y^x)^2 + (x_i^y - y^y)^2 + (x_i^z - y^z)^2)^{1/2}$$

Here $x_i = (x_i^x, x_i^y, x_i^z)$, $(i=1, 2, 3)$, $y = (y^x, y^y, y^z)$.

We will fix those initial positions of the pursuers which form a non-degenerate triangle. The evader is in one of the regions D_i $(i=1, 2, 3)$ (Fig. 1). The regions D_1 and D_2 exist in all cases of the mutual position of the players for any constraints on the control. The region D_3 always exists when the triangle formed by the pursuers is acute-angled.

We will introduce a system of coordinates such that x_1, x_2, x_3 lie in the XOY plane, and the axis of coordinates passes through the centre of the circle described around the triangle, forming a right triplet with XOY.

The region D_3 at the instant of time t is bounded by rotation around the axis of coordinates of an inner

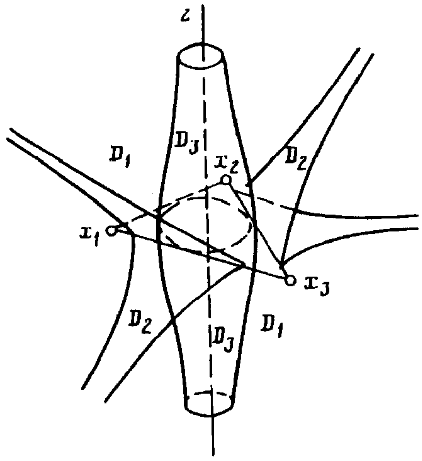

FIG. 1.

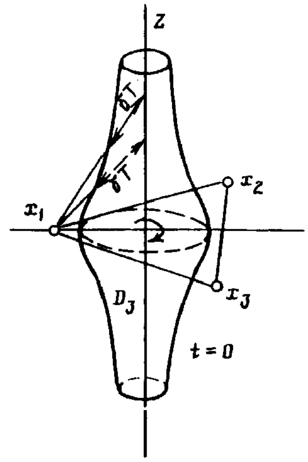

FIG. 2.

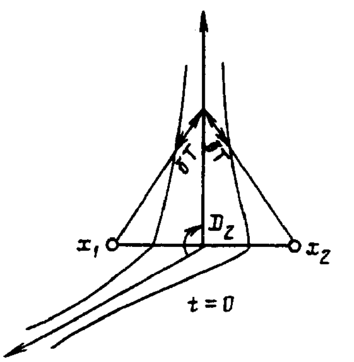

FIG. 3.

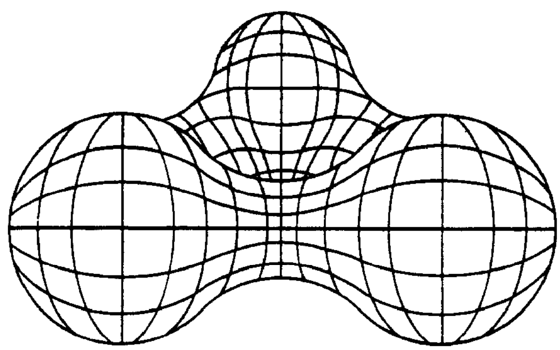

FIG. 4.

Nikomed concoid of radius $\sigma(T-t)$ relative to the point which coincides with arbitrary x_i $(i=1, 2, 3)$ and the axis of coordinates (Fig. 2).

To construct the region D_2 we will choose an arbitrary side of the triangle $x_1x_2x_3$, for example, $[x_1x_2]$. The part of the region D_2 is bounded by rotation about the side $[x_1x_2]$ of two inner Nikomed concoids of radius $\sigma(T-t)$ with respect to the middle of the perpendicular to $[x_1x_2]$ and correspondingly the points which coincide with x_1 and x_2. In a similar way, by choosing the remaining sides of the triangle we obtain the whole region D_2 (Fig. 3).

The part of the space not belonging to D_3 and D_2 belongs to D_1 (Fig. 1).

The singular manifold in region D_3 is characterized by the fact that the z coordinates of all the players are equal, and the singular manifold of region D_2 is characterized by the fact that the evader belongs to an arbitrary side of the triangle $x_1x_2x_3$, but in this case, unlike the problem considered earlier in [6], the evader has not one optimal motion but a continuum. Coincidence of the coordinates of the evader and the pursuer describes the singular manifold in region D_1 and here the optimal motion of the evader is not unique.

The programmed maximin function (2.4) can be written as follows:

$$\gamma(t, x_i) = \min_i \gamma_i(t, x_i) \quad (i=1, 2, 3)$$
$$\gamma(t, x_i) = ((y^z - x_i^z + (\sigma^2(T-t)^2 - (y^x)^2 - (y^y)^2)^{1/2})^2 + (x_i^x)^2 + (y^x)^2)^{1/2} - \rho(T-t) \quad (i=1, 2, 3)$$

We can check for system (1.1) and (1.2) that the Bellman–Isaacs equation is satisfied in the regular region D_3 and that the programmed maximin function is identical with the payoff function at the instant of completion.

Denoting the angles between the instantaneous speeds of players y and x_1 and the OX, OY and OZ areas, respectively by β^x, β^y, β^z and α_i^x, α_i^y, α_i^z we can rewrite the controls in the form

$$u_i^x = \rho\cos\alpha_i^x,\quad u_i^y = \rho\cos\alpha_i^y,\quad u_i^z = \rho\cos\alpha_i^z$$

$$v^x = \sigma\cos\beta^x,\quad v^y = \sigma\cos\beta^y,\quad v^z = \sigma\cos\beta^z$$

By choosing the angles as in Assertions 1 and 2 in accordance with (4.1), we can guarantee that there is no increase in the programmed maximin function. For the evader situated in D_3, we have $\alpha_1^z = \alpha_2^z = \alpha_3^z$.

The extremal aimings of all the players at one of the points of the set $\{y^*\}$ with maximal and constant controls are optimal strategies of all the players, and all the singular manifolds are dispersion surfaces.

The form of the level surface of the value of the game is shown in Fig. 4.

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